A Multi-factor Statistical Model for Interest Rates

Mark Reimers and Michael Zerbs

A term structure model that produces realistic scenarios of future interest rates is critical to the effective measurement of counterparty credit exposures. Scenarios are realistic when observed interest rates and actual exposures over time are consistent with the predicted distribution of interest rates and potential exposures. In this paper, we present a statistical term structure model with mean reversion. The model can be extended to term structures in several markets. A case study is used to explore the calibration, application and out-of-sample testing of the model in practice. The out-of-sample validation covers large, unanticipated changes in interest rates. In the case study, the model performs well in the estimation of potential exposures over longer time periods. The model may underestimate exposures in the short-term when shocks occur in the beginning of the out-of-sample testing period.

Financial institutions manage the credit risk of derivative portfolios by assigning credit lines to each counterparty name. Counterparty credit exposures measure the utilization of these credit lines at the counterparty level. **Credit exposure** is defined as the cost of replacing all contracts with a given counterparty at the time of default. Since default is an uncertain event that can occur at any time during the lives of the contracts, actual exposures to each counterparty today and potential changes in exposures over time must be managed. (Aziz and Charupat (1998) provide formal definitions for actual and potential exposures.) The ongoing management of credit lines can be effective only when potential exposures for the entire lives of the contracts are estimated correctly today, particularly in the case of derivative portfolios.

Potential exposures are frequently determined from a simulation of risk factors, such as interest rates, foreign exchange rates and equity prices. In the case of derivatives such as swaps, swaptions, caps or floors, the simulation typically covers 10 years or more. A term structure model that produces realistic scenarios of future interest rates is the foundation for the correct estimation of exposures. Scenarios are realistic when the distribution of future interest rates is close to the distribution of historical rates.

We present a statistical model with mean reversion for the evolution of the term structure of interest rates. The objective of this paper is to illustrate the practical implementation issues associated with calibrating, applying and testing the model.

A case study is used to explore these issues. The model is used to create scenarios on future rates that are then used to calculate the potential exposure of a test portfolio of swaps, swaptions, caps and floors. The scenarios are meaningful if they lead to accurate estimates of potential exposure. The model is calibrated to US Constant Maturity Treasury yields in a historical calibration period (January 1, 1984 to December 31, 1990). A second (non-overlapping) historical period (January 3, 1991 to December 31, 1998) is designated as the out-of-sample testing period. The out-of-sample test investigates whether the calibrated model produces realistic and meaningful interest rate scenarios by comparing the distribution of interest rates predicted by the calibration model to the historical outcome, and by comparing the potential exposures of the test portfolio based on these scenarios to the historical exposures realized in the validation period. Because changes in interest rates affect the exposures of long and short derivative positions differently, we also test the mirror image of the portfolio to investigate the sensitivity of both sides of the position.

The model is applied to determine the evolution of a single term structure. The Asset Block Decomposition technique (Reimers and Zerbs 1998) can be applied to extend the model to term structures in several markets. Using this decomposition, each term structure is defined as an asset block. A reduced set of principal components is defined for each asset block; the covariances between the components of paired blocks then link the components' movements.

The paper is organized as follows. We begin by describing the proposed term structure model and the calibration methodology. Next, we present the case study, beginning with the calibration results, followed by the results of the out-of-sample test. We close with an evaluation of the results and implications for further research.

A simulation model for interest rates

The basic elements of a long-term interest rate simulation model are a model for the joint fluctuations in the rates and some device to maintain the distributions of rates within bounds from the current time T_o, until the horizon time, T. A mean-reverting process is generally used to ensure that distributions remain bounded as time passes.

We assume that the term structure of interest rates is described by n rates. We further assume that joint movements of the logarithms of the n rates occur only within a k-dimensional subset of the n-dimensional directions, where k is smaller than n and the n directions are independent. A number of factor analysis techniques can be used to determine this subset of directions. In this paper, the directions are determined by Principal Component Analysis.

Let the term structure of interest rates be defined by a set of discrete rates, $r_i, i = 1, 2, \ldots, n$. We assume that as time evolves, each rate r_i reverts to a target value r_i^{∞}, which remains constant over time. Let $y_i = log(r_i)$ and $y_i^{\infty} = log(r_i^{\infty})$.

We assume there are k independent state variables, $x_1, x_2, \ldots, x_k$, $k < n$ that explain the movements of the log rates, y_i. Each state variable follows an Ornstein-Uhlenbeck process (Karatzas and Shreve 1994):

$$dx_j = -a_j x_j dt + \sigma_j dz_j \qquad j = 1, 2, \ldots, k \tag{1}$$

where a_j specifies the mean-reversion speed, σ_j specifies the instantaneous volatility and dz_j represents a random fluctuation of the associated state variable, specifically, a Brownian motion. The volatilities and the mean-reversion rates are constant through time.

The log rates are reconstructed by

$$y_i = y_i^{\infty} + \sum_{j=1}^{k} b_{ij} x_j \qquad i = 1, 2, \ldots, n \tag{2}$$

where $\mathbf{B} = [b_{ij}]$. The columns, $\mathbf{B}_j$, of the matrix $\mathbf{B}$ are orthonormal vectors. Here, the columns of $\mathbf{B}$ are the first k principal components of the changes in log rates.

The individual rates are given by

$$r_i = e^{y_i} \tag{3}$$

Note that Equations 1 to 3 bear some resemblance to the Black-Karasinski model for derivatives pricing (Black and Karasinski 1991). The most evident differences are that this model is based on several principal components, the short rate plays no special role and the model is "market describing" rather than "market fitting" (Rebonato 1996, p. 337). Because it is not fitted to today's term structure of interest rates or implied volatilities, the model is not a no-arbitrage pricing model as are the Hull-White (Hull and White 1990) or Heath-Jarrow-Morton

(Heath et al. 1989) approaches. A market price quoted today cannot be recovered from the model, however, the pricing of derivatives is not a modeling objective here. The model also differs from most equilibrium models in that it is not derived from a stylized version of the economy as are the Cox-Ingersoll-Ross (Cox et al. 1985) or Longstaff-Schwartz (Longstaff and Schwartz 1992) models.

An important feature of the interest rate model described in Equations 1 to 3 is that the variance of all market rates at all future times, $T_o \le t \le T$, can be calculated by a closed formula. The variance of the state variable x_j at time t is

$$\sigma_{x_j}^2(t) = \frac{\sigma_j^2}{2a_j}(1 - e^{-2a_j(t - T_o)}) \tag{4}$$

Therefore, the variance of the market rate y_i at time t is

$$\sigma_{y_i}^2(t) = \sum_{j=1}^{k} \frac{\sigma_j^2}{2a_j}(1 - e^{-2a_j(t - T_o)})b_{ij}^2 \tag{5}$$

Since the distribution of each log rate is normal, quantiles for both the log rates, y_i, and hence for the rates, r_i, can be determined for all t.

Calibration

The interest rate simulation model (Equations 1 to 3) is completely specified when the significant principal components and thus the state variables x_j and the matrix $\boldsymbol{B}$ are determined, when values for the mean-reversion parameters a_j and variances σ_j^2 associated with those state variables are established and when a target value y_i^∞ for each node i on the term structure is set. The purpose of the model calibration is to estimate these values.

Calibration methodology

We describe a calibration methodology that relies on historical data. The starting point for the calibration is historical observations for each interest rate r_i over the calibration period. The calibration period $[t_o, T]$ is divided into M equally distributed observations denoted by $t_m, m = 1, 2, \ldots, M$. The historical observations for each interest rate are referred to as $r_i(t_m)$. This time series of historical interest rates is transformed into a time series of log rates, $y_i(t_m)$, using Equation 3. The target values, y_i^∞, are calculated as the sample means of the log rates over the calibration period:

$$y_i^\infty = \frac{1}{M}\sum_{m=1}^{M} y_i(t_m) \tag{6}$$

Equation 3 then determines r_i^∞ given y_i^∞.

The state variables x_j and the matrix $\boldsymbol{B}$ are the result of a Principal Component Analysis. Defining the elements of $\boldsymbol{B}$ as the principal components of changes in the log rates y_i ensures that the state variables are independent of each other.

The historical values for the state variable x_j over time are referred to as **implied state variable histories,** $x_j(t_m)$. As before, m indexes the observations in the calibration period $[t_o, T]$. Because the columns of the matrix $\boldsymbol{B}_j$ are orthonormal vectors, Equation 2 can be rewritten as

$$x_j(t_m) = \sum_{i=1}^{n} b_{ji} \times (y_i(t_m) - y_i^\infty) \tag{7}$$

A proof is presented in the Appendix.

Given $x_j(t_m)$, we obtain a time series of state variable changes, $dx_j(t_m)$ according to

$$dx_j(t_m) = x_j(t_m) - x_j(t_{m-1}) \tag{8}$$

for $m = 2, 3, \ldots, T$.

Assuming $a_j = 0$, then from Equation 1 $dx_j = \sigma_j dz_j$, that is, the variance σ_j^2 can then be estimated directly as a sample variance from the state variable changes $dx_j(t_m)$:

$$\sigma_j^2 = \frac{1}{M-2} \times \sum_{m=2}^{M} dx_j(t_m)^2 \qquad (9)$$

We can conclude the calibration by first estimating the sample variance of state variable levels from the implied state variable histories $x_j(t_m)$, using this as an approximation for $\sigma_{x_j}^2(t)$ in Equation 4, then solving Equation 4 to obtain the mean-reversion parameters a_j .

Calibration results

In this section we present the results of the model calibration, including the determination of the principal components, the mean-reversion parameter and the target.

The calibration period $[t_o, T]$ is the seven-year period from January 1, 1984 to December 31, 1990. Historical data for US Constant Maturity Treasury Yields (US Federal Reserve Board 1999) is used in the calibration. The term structure is defined by nine key rates: three and six months, one, two, three, five, seven, 10 and 30 years. Tenors shorter than three months are omitted because their high volatility makes Principal Component Analysis problematic.

Principal Component Analysis

The first three principal components (PC) of the daily log rates over the term structure are depicted in Figure 1and summarized in Table 1.

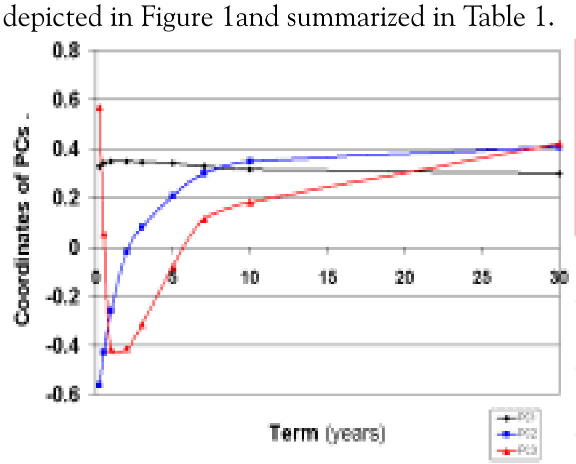

Figure 1: Principal components of daily log rates

Terms	PC 1	PC 2	PC 3
3mo	0.3626	–0.49378	0.60272
6mo	0.3743	–0.39229	0.10857
1yr	0.3798	–0.26574	–0.40852
2yr	0.3585	–0.02328	–0.41570
3yr	0.3434	0.08809	–0.33622
5yr	0.3178	0.25442	–0.02507
7yr	0.2993	0.33271	0.09108
10yr	0.2835	0.38263	0.15922
30yr	0.2585	0.44893	0.37140

Table 1: Principal components of the daily log rates

The first principal component of the daily log rates represents an (almost) parallel shift of the yield curve. The long rates in the US term structure have lower volatilities than the short rates, and hence their loadings on this "shift" component are slightly lower. The second principal component represents a "twist" of the yield curve; the long rates move against the short rates. The third component represents a "butterfly" movement; the two-, three- and 5-year rates move against both the long and the short end of the curve.

Table 2 presents the distribution among the first three principal components of the variance of all nine log rates during the calibration period. The first principal component explains 93.03% of the total variance, and together, the first three components explain 99.89% of the total variance of the daily log rates.

Accordingly, we conclude that during the calibration period, three principal components are sufficient to describe the variance of the daily log rates. Thus, these three principal components become the three state variables of the model, $k = 3$ and the entries of Table 1 are the values of the matrix $\boldsymbol{B}$.

Principal Component	Percentage of total variance	Cumulative percentage of total variance
PC 1	93.03	93.03
PC 2	6.56	99.59
PC 3	0.30	99.89

Table 2: Distribution of variance of daily log rates, σ_j^2 (%)

Mean-reversion parameter

The time series of daily changes for the state variables identified in the Principal Component Analysis, $dx_j(t_m)$, calculated according to Equation 8, are shown in Figure 2. The implied history for the shift component peaks shortly after the start of the calibration period, then decreases in the first half of the calibration period and resumes a weak upward trend for the second half of the period. The second and third components oscillate around the target.

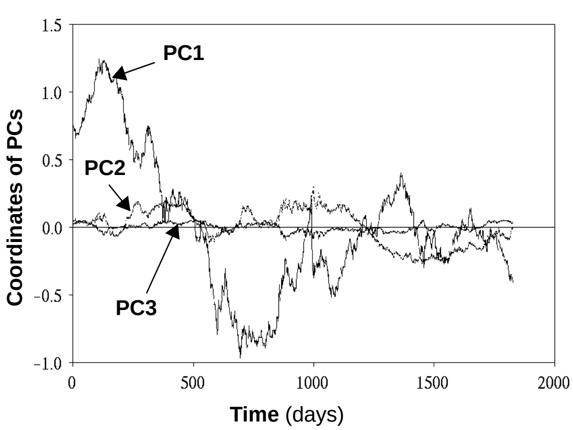

Figure 2: Histories of state variable changes over the calibration period

The mean-reversion rates are set to zero and the variances of the state variable daily changes, σ_j^2, calculated according to Equation 9.

The iterative calibration methodology is applied to determine the mean-reversion rates given the variances of the principal components. The resulting estimates of annualized mean-reversion rates for each principal component are presented in Table 3. The first two mean-reversion rates are deemed to be close enough to zero to accept; the third is different than zero, but the impact of the third principal component is not significant. Thus, all three estimates are adopted.

Principal Component	Mean-reversion rate (annualized) a_j
PC 1	0.001
PC 2	0.066
PC 3	2.120

Table 3: Estimated annualized mean-reversion rates

Target

The target values for the testing period, y_i^∞, are calculated as the sample means of the log rates over the calibration period according to Equation 6.

Out-of-sample testing

The calibrated model is used to generate scenarios over an out-of-sample, historical testing period from January 3, 1991 to December 31, 1998. The actual and potential exposures for a test portfolio of swaps, swaptions, caps and floors are estimated based on these scenarios.

Historical data for US Constant Maturity Treasury Yields (US Federal Reserve Board 1999) is used in the testing. The term structure for the actual rates and the target rates are shown in Figure 3. Actual rates range from 6.66% in the short end to 8.14% in the long end of the curve. The target curve resulting from the calibration is more than 100 basis points higher than the actual curve in early 1991.

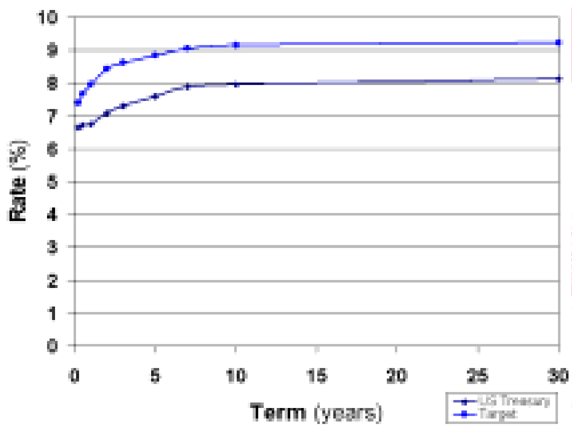

Figure 3: Actual Treasury and target curves

The scenarios generated are validated using a three-phase test. First, the distribution of the interest rates predicted by the calibrated model is compared to the historical outcome of interest rates observed in the out-of-sample testing period to determine if the model provides realistic scenarios.

Next, an out-of-sample test is performed to determine the ability of the model to generate meaningful scenarios that lead to reasonable estimates of potential exposure. The scenarios generated by the calibrated model are used to compute future values for all positions in a test portfolio, under each scenario at each time step. The distribution of these Mark-to-Future values is used to compute worst-case potential exposures at the 95% confidence level, over the out-of-sample period. The worst case potential exposures result in an envelope of potential exposures. The historical actual exposures realized during the out-of-sample testing period are compared to the estimated envelope of potential exposures. If the scenarios generated fall outside of the envelope, the model is likely inappropriate.

Finally, because the impact of a change in rates affects the exposure of long and short derivative positions differently, the exposures of the mirror image of the portfolio are studied to test the sensitivities of both sides of the positions.

Testing the scenarios generated

A preliminary test of the realism of the model compares the interest rates actually observed through the out-of-sample testing period with the distribution predicted by the calibrated model.

A convenient summary of the distribution at any time is the central 95% inclusion range. Equation 4 is used to calculate the 2.5th and 97.5th percentiles of the distribution of the interest rates at each point in time. The curves created by linking these forecasts through time define the **inclusion envelope**.

Example results are presented in Figure 4 which shows the historic values and the lower bound of the 95% inclusion envelope for the 30-year rates. Results for other rates are similar. Note that the historic rates are generally greater than the lower bound of the envelope. For all nine rates, the historical data falls outside the 95% inclusion envelope for the calibrated model on 7.7% of days observed, indicating that the scenarios generated by the model are realistic.

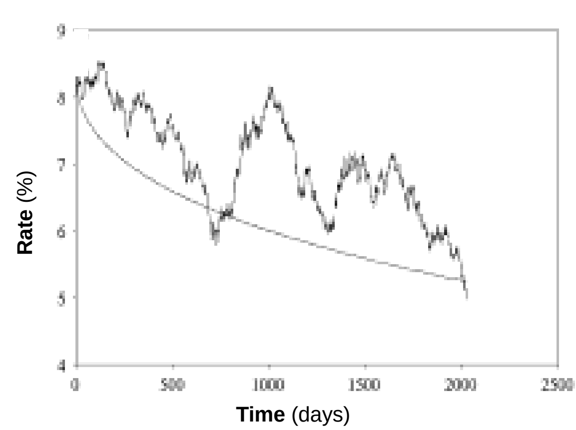

Figure 4: Historic rates and the lower bound of the 95% envelope for the 30-year rate

Testing potential exposures

The exposure functions of derivative portfolios are non-linear across scenarios and over time. Hence, a study of exposures on a test portfolio of interest rate derivatives may yield different results than a test of the interest rate scenarios themselves.

The test portfolio comprises at-the-money fixed-floating swaps covering a range of tenors from one to 10 years, a swaption on a 10-year bond, and several long term caps and floors that are at-

Name	Position	Fixed Coupon / Strike (interest rate p.a.)	Maturity (days)	Value (USD)
Cap (ATM)	1	7.50	3653	4,904,552
Cap (far OTM)	–1	11.00	3653	–477,731
Cap (OTM)	–1	9.00	3653	–1,848,424
Floor (ATM)	1	7.10	3653	2,661,553
Floor (far OTM)	–1	5.43	3653	–193,317
Floor (OTM)	–1	6.20	3653	–686,447
Swaption (10yr)	1	7.95	90	1,091,133
1y swap	–1.8	7.00	365	–80,955
2y swap	1	7.20	731	–78,341
3y swap	–1.55	7.40	1096	69,727
5y swap	1	7.68	1826	23,312
7y swap	–2	7.90	2557	–93,044
10y swap	1	7.95	3653	41,878
Portfolio Total	N/A	N/A	N/A	5,333,895

Table 4: Portfolio holdings

the-money (ATM), out-of-the-money (OTM), and far out-of-the money. These positions are designed to be sensitive to very large interest rate movements. The portfolio is constructed to have exposures that are partially offsetting. All positions are denominated in USD, have a notional amount of 100 million USD and all are priced using the same term structure (Figure 3). The portfolio is held by a single counterparty. To allow for full close-out netting, all positions are subject to the same master agreement. A detailed description of the portfolio holdings is given in Table 4.

Equations 1 to 3 are used to generate 1,000 scenario paths over time for the three state variables identified in the calibration period. The paths reach seven years into the future, with quarterly time steps in the first year, and semi-annual time steps thereafter.

In each scenario and at each time step, the future value of each position is computed and placed in a table of Mark-to-Future values. The values are aggregated at the portfolio level. Figure 5 illustrates the evolution of the mean of the Mark-to-Future values and of the upper and lower limits of the range between which the Mark-to-Future values fall with 95% confidence.

Given the Mark-to-Future values, we calculate potential exposure under each scenario and compare the predicted distribution with the actual historical outcomes in the out-of-sample testing period. The mean potential exposure, the portfolio exposure at the 95% confidence level and the actual exposure of the portfolio are shown in Figure 6. Note that only non-negative Mark-to-Future values contribute to potential exposure. The portfolio exposure doubles from 10 million USD at the beginning of the simulation to more than 20 million USD near the end of the simulation. The estimated potential exposure of the portfolio consistently dominates its actual exposure.

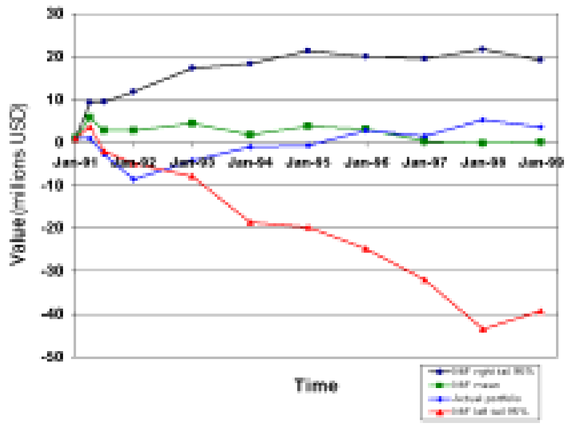

Figure 5: Portfolio Mark-to-Future values

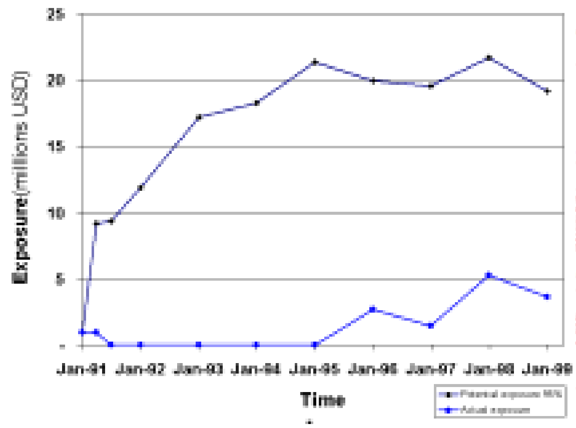

Figure 6: Out-of-sample test of potential exposures

Table 5 presents actual exposures as a percentage of potential exposures at the 95% confidence level for each position. There is only one excess exposure: the actual exposure of the 2-year swap exceeds the potential exposure by 12% at the one year point. The results for caps and floors are presented in the aggregate; there are no excess exposures. At the portfolio level, actual portfolio

	Future time (m/d/y)									
Name	4/1/91	7/1/91	1/2/92	1/4/93	1/3/94	1/3/95	1/3/96	1/3/97	1/3/98	1/3/99
Caps	53	45	14	7	2	10	0	1	0	0
Floors	86	61	96	63	66	23	42	38	50	68
Swaption (10y)	4									
1y swap	0									
2y swap	10	52	112							
3y swap	0	0	0	0	0					
5y swap	0	14	86	57	61	12	72			
7y swap	1	0	0	0	0	0	0	0		
10y swap	0	3	63	41	54	4	56	27	36	50
Portfolio Total	10	0	0	0	0	0	14	8	24	19

Table 5: Actual versus potential exposures (%)

exposures reach a peak of 24% of potential exposures in 1998. These results offer evidence that the model generates realistic estimates of worst case exposures.

The results summarized in Table 5 indicate that actual exposures as a percentage of potential exposures for short positions, such as the 7-year swap, are much lower than those for long positions, such as the 10-year swap.Testing both sides of a position

To assess whether the results of the test change if the long and short positions in the portfolio are reversed, we apply the calculations in the out-of-sample test to the original portfolio and its mirror image. Specifically, we compare the envelope of Mark-to-Future values at the 95% confidence level to the absolute value of the actual outcomes over time. Figure 7 graphs the results for the aggregated portfolio. Actual outcomes exceed the lower tail at the first three time points. At all subsequent points, actual outcomes are well within the envelope.

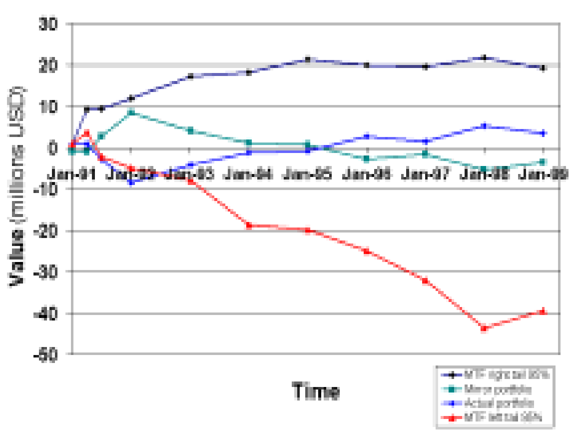

Figure 7: Portfolio Mark-to-Future values

To facilitate the review of the extended out-of-sample test at the position level, we define three supporting measures:

- **Envelope of differences**: the difference between the position's median Mark-to-Future value and its potential future value at the 97.5th or 2.5th percentile at each time step.
- **Actual difference**: the difference between the position's actual market value and the median of its Mark-to-Future values at each time step.
- **Envelope utilization**: the absolute value of the position's actual difference, expressed as a percentage of its envelope of differences.

An envelope utilization of more than 100% indicates that actual market values exceed Mark-to-Future values at the 95% envelope. As we test for envelope utilizations of more than 100% at the upper and lower percentile, we also account for situations where long and short positions are reversed.

Table 6 presents the envelope utilizations for each position and for the entire portfolio over the out-of-sample test period. Outcomes outside the 95% envelope are highlighted in light grey. Outcomes that also fall outside the 99% envelope are highlighted in dark grey.

At the position level, about 10% of the outcomes (8/75) fall outside the 95% envelope. The envelope utilizations exceed 100% for the floors, for the short-dated swaps and for the entire portfolio. Excess utilizations at the position level are restricted to the first four steps of the out-of-sample test. Four out of eight excesses also exceed the 99th percentile, but all four fall in the first time step.

Are the observed excess utilizations consistent with the actual interest rates for the period 1991 to 1993? Money market rates fell by more than 70 basis points in the first three months of 1991. By 1992 rates had dropped by more than 250 basis points in the short end and by about 100 basis points in the long end. This steep decline in interest rates contrasts with a target curve that is above the market rates on the valuation date (Figure 3)! The calibrated model does not produce scenarios that reflect these extreme rate changes in the first year of the simulation. Hence, we should expect some excess envelope utilizations in the out-of-sample test.

	Future Time (m/d/y)									
Position	4/1/91	7/1/91	1/2/92	1/4/93	1/3/94	1/3/95	1/3/96	1/3/97	1/3/98	1/3/99
Caps	10	16	58	62	19	45	26	42	54	59
Floors	200	176	138	135	43	8	77	36	23	33
Swaption 10y	91	14	59	39	53	7	5	33	45	65
1y swap	62	404	930	-	-	-	-	-	-	-
2y swap	14	21	119	-	-	-	-	-	-	-
3y swap	3	8	110	41	-	-	-	-	-	-
5y swap	8	12	84	54	57	7	38	-	-	-
7y swap	7	13	69	44	58	10	60	18	-	-
10y swap	7	14	59	39	53	7	57	33	45	65
Portfolio Total	196	112	142	70	5	7	23	30	50	47

Table 6: Envelope utilizations (%)

Short-dated swaps are sensitive only to changes in the short end of the yield curve. We also expect that unrealized gains and losses in the short positions in the out-of-the-money floors change rapidly as these positions become at-the-money or in-the-money options. It follows that a sharp decline in interest rates affects these holdings more than it affects the caps or the long-dated swaps. Our tests show that even a small error in forecasting the values of the risk factors can lead to a large error in forecasting the exposure associated with a position. As these positions are designed to be sensitive to very large interest rate movements, we are not surprised to observe that about 10% of potential exposures fall outside of the envelope, while only 8% of the forecasted interest rates fall outside the envelope.

Nevertheless, these results are encouraging. The out-of-sample testing period covers a dramatic fall in interest rates between 1991 and 1992 and a dramatic increase in rates between 1994 and 1995. In spite of this, actual exposures at the portfolio level are always less than estimated potential exposures (Table 5). Outcomes that are outside the envelope are limited to the first four time steps of the out-of-sample test, regardless of whether the position is long or short. The case study also illustrates that portfolio offsets reduce the potential for actual exposures to exceed estimated worst case exposures (Table 6).

Conclusions

We have presented a statistical model for generating interest rate scenarios over time. The objective of the model is the estimation of counterparty exposures. The model provides a concise description of interest rate dynamics and is calibrated easily to historical data. It can be extended to term structures in different markets.

To test whether the model can be used in the effective management of potential credit exposures, we have calibrated the model to US rates and performed an out-of-sample test on the scenarios and exposures generated by the model. We observe that

- The model performs well when estimating potential exposures over longer time periods that include extreme market moves such as the sudden rise in interest rates in 1994.
- The model may underestimate exposures for the short-term when extreme market moves occur in the beginning of the out-of-sample testing period.
- Portfolio offsets tend to mitigate the severity of exposure excesses.
- The potential exposure of the test portfolio is congruent with the distribution of interest rate scenarios.

Useful further work would include the calibration and out-of-sample testing of the model for other term structures, testing for sensitivity to the model parameters and calibration assumptions and a comparison of the results to those derived from other models. Proper backtesting of long-term simulations in general and of credit risk in particular is problematic because of the quantity of time series data required. Further work might also include investigation of backtesting methodologies.

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United States Federal Reserve Board Web site: http://www.bog.frb.fed.us/Releases/H15/ then select historical data for July 6, 1999.

Appendix

Rewriting Equation 2 in vector notation

$$y = y^{\infty} + \sum_{j=1}^{k} \mathbf{B}_j x_j \qquad \text{(A1)}$$

Equation A1 is pre-multiplied by column vector $\mathbf{B}_{j_o}^T$, where j_o indexes any column of $\boldsymbol{B}$, $1 \le j_o \le k$:

$$\mathbf{B}_{j_o}^T(y - y^{\infty}) = \mathbf{B}_{j_o}^T \sum_{j=1}^{k} \mathbf{B}_j x_j = \sum_{j=1}^{k} \mathbf{B}_{j_o}^T \mathbf{B}_j x_j = x_{j_o}$$

Because the columns of $\boldsymbol{B}$ are orthonormal

$$\mathbf{B}_{j_o}^T \mathbf{B}_j = 1 \qquad if \qquad j_o = j$$

$$\mathbf{B}_{j_o}^T \mathbf{B}_j = 0 \qquad otherwise$$

Therefore,

$$x_j = \mathbf{B}_j^T(y - y^{\infty}) \qquad j = 1, 2, \ldots, k$$