Modeling Credit Risk with Partial Information

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Abstract

This paper provides an alternative approach to Duffie and Lando [7] for obtaining a reduced form credit risk model from a structural model. Duffie and Lando obtain a reduced form model by constructing an economy where the market sees the manager's information set plus noise. The noise makes default a surprise to the market. In contrast, we obtain a reduced form model by constructing an economy where the market sees a reduction of the manager's information set. The reduced information makes default a surprise to the market. We provide an explicit formula for the default intensity based on an Azéma martingale, and we use excursion theory of Brownian motions to price risky debt.

KEYWORDS: Default risk, Azéma martingale, Brownian excursions, default distribution.

1 Introduction

Reduced form models have become important tools in the risk management of credit risk (for background references see Jarrow and Yu [10] and Bielecki and Rutkowski [1]). One reason for this is that they usually provide a better fit to market data than structural models do (see Jones, Mason, Rosenfeld [12], Jarrow, van Deventer, Wang [11], Eom, Helwege, Huang [9]). Reduced form models take a firm's default process as exogenous with the time of default an inaccessible stopping time. This implies that the market cannot predict the

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time of default. Yet, managers working within a firm surely know when default is imminent. From a manager's perspective, default is an accessible stopping time (predictable). Usually, in the structural approach default occurs when the firm's value, a continuous sample path process, hits a barrier. This formulation is consistent with the manager's perspective but inconsistent with reduced form models.

Duffie and Lando [7] link the two perspectives by introducing noise into the market's information set, transforming the manager's accessible default time from the structural approach into the market's inaccessible default time of a reduced form model. Duffie and Lando postulate that the market can only observe the firm's asset value plus noise at equally spaced, discrete time points (and not continuously). And, when default occurs, the market is immediately informed. This noise generates the market's surprise with respect to default, because the firm could nearly be in default (just about to hit the barrier) and the market not yet aware of its imminence. Kusuoka [13] extends Duffie and Lando's model to continuous time observations of the firm's asset value plus noise. Kusuoka's solution is an application of continuous time filtering theory.

This approach to constructing a reduced form credit model presumes that the market has the same information set as the firm's management, but with noise appended.[1] An interpretation is that accounting reports and/or management press releases either purposefully (e.g. Enron) or inadvertently add extraneous information that obscures the market's knowledge of the firm's asset value. Management knows the firm's value (because this knowledge determines default), but they cannot (or will not) make it known to the market. The market's task is to remove this extraneous noise. Although possible in many situations, this characterization of management's information versus the market's is not exhaustive. An alternative and equally plausible characterization is that the market has the same information as a firm's management, but just less of it. Accounting reports and/or management press releases provide just a reduced set of the information that is available.

Consistent with this alternative perspective, we provide a second approach to the construction of a reduced form credit risk model from a structural model. In our approach, the firm's cash flows, a continuous sample path process, provide the sufficient statistic for default. If the firm's cash flows remain negative for an extended period of time, the firm after exhausting both its lines of credit and easily liquidated assets, defaults. Management observes the firm's cash flows. In contrast, the market observes only a very coarse partitioning of the manager's information set. The market knows only that the cash flow is negative - the firm is experiencing financial distress - and the duration of the negative cash flow event - nothing else. This information structure has default being an

[1]Filtering theory was originally formulated for electronic signal processing where the physical problem corresponds to a situation where an electronic signal is received with noise and the noise needs to be "filtered" out.

accessible stopping time for management, but an inaccessible stopping time for the market, yielding the reduced form credit risk model.

To illustrate the economic concepts involved, this paper concentrates on developing a specific example to obtain analytic results. The analytic results solidify intuition and make the economic arguments more transparent. Generalizations and extensions will be readily apparent once the example is well understood. It is our hope that this paper will motivate additional research into this area. Our example provides an explicit representation of the firm's default intensity using an Azéma's martingale (see Emery [8]). To illustrate the usefulness of this result, we compute the value of a risky zero-coupon bond using excursion theory of Brownian motions. For another application of excursion theory to option pricing see Chesney, Jeanblanc-Piqué, Yor [3].

An outline for this paper is as follows. Section 2 presents the structural model. Section 3 presents the reduced form model, section 4 values a risky zero-coupon bond in the reduced form model, while section 5 concludes the paper.

2 The Structural Model

We consider a continuous trading economy with a money market account where default free zero-coupon bonds are traded. In this economy there is a risky firm with debt outstanding in the form of zero-coupon bonds. The details of these traded assets are not needed now, but will be provided later as necessity dictates. The market for these traded securities is assumed to be arbitrage free, but not necessarily complete.

We begin with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q})$ satisfying the usual conditions. Time $T > 0$ is the final date in the model. The probability $\mathbb{Q}$ is an equivalent martingale probability measure under which the normalized prices of the traded securities follow a martingale. Normalization is by the value of the money market account. The no arbitrage assumption guarantees the existence, but not the uniqueness of such a probability measure (see Duffie [6]).

2.1 Management's Information

Let X be the cash balances of the firm, normalized by the value of the money market account, with the following stochastic differential equation:

$$dX_t = \sigma dW_t, \ X_0 = x \tag{1}$$

with $x > 0$, $\sigma > 0$, and where W is a standard Brownian motion on the given probability space.

The cash balances of the firm are initialized at $x > 0$ units of the money market account. One should interpret this quantity as the "target" or "optimal" cash balances for the firm. An optimal cash balance could exist because if the firm holds too much cash, it foregoes attractive investment projects and incurs increased tax liabilities, while if it has too little cash, it increases the likelihood of bankruptcy and the occurrence of third party costs (see Brealey and Myers [2] for related discussion). The firm attempts to maintain cash balances at this target level, but fluctuations occur due to its operating needs, e.g. meeting payrolls, paying suppliers, receiving payments from accounts receivable, etc. However, without loss of generality, to simplify the presentation we assume that $x = 0$.

Under the martingale measure, cash balances have no drift term. Under the empirical measure, however, one would expect that the cash balances should drift at the spot rate of interest. This is consistent with the firm holding its cash balances in the money market account and trying to maintain the target level balance.

The firm's management observes the firm's cash balances. Cash balances can be positive, zero, or negative. Negative cash balances correspond to situations where payments owed are not paid, and the firm is in financial distress.

2.2 The Default Process

Let $\mathcal{Z} := \{t \in [0,T] : X(t) = 0\}$ denote the times when the firm's cash balances hit zero. When the cash balances hit zero, the firm has no cash left for making current payments owed. The firm is in financial distress. With zero or negative cash balances, debt payments can only be made by liquidating the firm's assets or by accessing bank lines of credit. The firm can exist with negative cash balances for only a limited period of time. If the cash balances remain negative for an extended period of time, then default occurs. We now formalize this default process.

Associated with the zero set, we define the following function:

$$g(t) := \sup\{s \leq t : X_s = 0\}.$$

The random time $g(t)$ corresponds to the last time (before t) that cash balances hit zero. Let

$$\tau_\alpha := \inf\{t > 0 : t - g(t) \geq \frac{\alpha^2}{2} \text{ where } X_s < 0 \text{ for } s \in (g(t-), t)\}$$

for some $\alpha \in \mathbb{R}_+$ be the random time that measures the onset of a potential default situation for the firm. Formally, τ_α is the first time that the firm's cash balances have continued to be negative for at least $\alpha^2/2$ units of time. The constant α is a parameter of the default process (that could be estimated from market data). After time τ_α, if the firm's cash balances continue to be negative and also increase in absolute value, default occurs. We let τ_d denote the time of default. We assume that

$$\tau_d := \inf\{t > 0 : X_t = 2X_{\tau_\alpha}, \quad X_s < 0 \text{ for } s \in (\tau_\alpha, t)\}.$$

Default occurs the first time, after τ_α, that the cash balances remain negative and double in absolute magnitude. The intuition is that after this magnitude of unpaid balances, the lines of credit have been drawn down, and the firm can no longer meet its debt payments. The doubling in absolute magnitude of the cash balances prior to default is only for analytic convenience, and it has no economic content. The generalization of this assumption is a subject for future research. The above process is what the firm's management observes.

3 The Reduced Form Model

This section studies the structural model under the market's information set. It is shown here that the bankruptcy process, as viewed by the market, follows a reduced form model where the indicator function of the default time is a point process with an intensity.

In contrast to the manager's information, the market does not see the firm's cash balances. Instead, the market only knows when the firm has positive cash balances or when they have negative or zero cash balances. In the latter case, the market knows the firm is in a financially distressed situation.

We define

$$sign(x) = \{1 \text{ if } x > 0, \ -1 \text{ if } x \leq 0\}.$$

Set $\tilde{\mathcal{G}}_t := \sigma\{sign(X_s); s \leq t\}$ and let $(\mathcal{G}_t)_{0 \leq t \leq T}$ denote the $\mathbb{Q}$-complete and right continuous version of the filtration $(\tilde{\mathcal{G}}_t)_{0 \leq t \leq T}$. $(\mathcal{G}_t)_{0 \leq t \leq T}$ is the information set that the market observes. As seen, the market's information set is a very coarse filtering of the manager's information set. In essence, the market observes when the market is in financial distress, and the duration of this situation $(t - g(t))$.

Given this information, the market values the firm's liabilities by taking conditional expectations under the martingale measure $\mathbb{Q}$. This valuation is studied in the next section. For subsequent usage, note that $X_t = \sigma W_t$ so that the $\text{sign}(X_s) = \text{sign}(W_s)$ and the zero set of X is the same as that of W.

We now derive the intensity for the default time as seen by the market. First, let

$$\tau = \inf\{t > \tau_\alpha : X_t = 0\}.$$

Given the strong Markov property of X under $\mathbb{Q}$, and using the reflection principle, we see that τ and τ_d have the same distribution. The analysis in the rest of this section applies to τ. Let $\tilde{X}_t = \frac{1}{\sigma}\frac{2}{\sqrt{\pi}}X_t$. Then signs and zero sets of X and $\tilde{X}$ are the same.

Define $M_t := E[\tilde{X}_t|\mathcal{G}_t]$. Then, M is the Azéma's martingale on $(\Omega, (\mathcal{G}_t)_{0 \leq t \leq T}, \mathbb{Q})$.[2] Its quadratic variation satisfies the following "structure equation":

$$d[M, M]_t = dt - M_{t-}dM_t. \tag{2}$$

Azéma's martingale is a strong Markov process. For an extensive treatment of Azéma's martingale and the structure equation, see Emery [8]. We also have

[2]Note that Azéma's martingale has already been used in finance, but in a diferent context; see Dritschel and Protter [5].

the following formula for M:

$$M_t = sign(X_t)\sqrt{2}\sqrt{t-g_t}. \tag{3}$$

It is easily seen that τ can be equivalently written as

$$\tau = \inf\{t > 0 : \Delta M_t \geq \alpha\}.$$

Therefore, τ is a jump time of the Azéma's martingale, hence it's totally inaccessible in the filtration $(\mathcal{G}_t)_{0\leq t\leq T}$. Also note that $\tau_\alpha = \inf\{t > 0 : M_{t-} \leq -\alpha\}$.

But $\Delta M_t = -M_{t-}1_{[M_{t-}\neq M_t]}$. So, $\tau_\alpha \leq \tau$ a.s. Furthermore, τ_α is a predictable stopping time which implies $Q[\tau = \tau_\alpha] = 0$. Hence, $\tau_\alpha < \tau$ a.s.

Define $N_t := 1_{[t\geq\tau]}$. By the Doob-Meyer decomposition (see, e.g. Protter [15], pp. 90), there exists a continuous, increasing, and locally natural process, A, such that $N - A$ is a $\mathcal{G}$-martingale which has only one jump, at τ, and of size equal to 1.

Theorem 1 *τ has a $\mathcal{G}$-intensity, that is A is of the form $A_t = \int_0^{t\wedge\tau}\lambda_s ds$. Furthermore, $\lambda_t = 1_{[t>\tau_\alpha]}\frac{1}{2[t-g(t-)]}$ for $t \in [0,\tau]$.*

Proof. Let $A_t = A_{t\wedge\tau}$. Then

$$H_t := N_t - A_t = \int_0^{t\wedge\tau} h_s dM_s \tag{4}$$

for some $\mathcal{G}$-predictable process h_s since M possesses the predictable representation property. (This is proved in Emery [8].) Since A is continuous and of finite variation, and N is a quadratic pure jump semimartingale, we have

$$[H,H]_t = [N,N]_t = N_t. \tag{5}$$

Also,

$$[H,H]_t = \int_0^{t\wedge\tau} h_s^2 d[M,M]_s = \int_0^{t\wedge\tau} h_s^2 ds - \int_0^{t\wedge\tau} h_s^2 M_{s-}dM_s, \tag{6}$$

where the second equality follows from (2). Combining (4), (5) and (6) yields

$$\int_0^{t\wedge\tau} h_s^2 ds - \int_0^{t\wedge\tau} h_s^2 M_{s-}dMs - A_t = \int_0^{t\wedge\tau} h_s dM_s,$$

which implies

$$\int_0^{t\wedge\tau} h_s^2 ds - A_t = \int_0^{t\wedge\tau} h_s^2 M_{s-}dMs + \int_0^{t\wedge\tau} h_s dM_s. \tag{7}$$

The left side of the previous expression is continuous. Hence

$$\int_0^{t\wedge\tau}(h_s^2 M_{s-} + h_s)dM_s = 0. \tag{8}$$

We compute the predictable quadratic variation to get

$$\int_0^{t\wedge\tau} h_s^2(h_s M_{s-}+1)^2 ds = 0. \tag{9}$$

The Optional Sampling Theorem implies that $(N-A)_{t\wedge\tau_\alpha} = 0$, since $N=0$ before and at τ_α. Therefore we get $h=0$ on $[0,\tau_\alpha]$. On the other hand (9) gives $h_s = 0$ or $h_s = -\frac{1}{M_{s-}}$ on $[\tau_\alpha,\tau]$. But (4) implies h_s cannot be identically 0 on $(\tau_\alpha,\tau]$, and we see that $h_s = -1_{[s>\tau_\alpha]}\frac{1}{M_{s-}}$ satisfies (8). Therefore we deduce a version of H which is given by

$$H_t = -\int_0^{t\wedge\tau} 1_{[s>\tau_\alpha]}\frac{1}{M_{s-}}dM_s$$

and thus H jumps only at τ and its jump size is given by

$$\Delta H_\tau = -1_{[\tau>\tau_\alpha]}\frac{1}{M_{\tau-}}\Delta M_\tau = -\frac{1}{M_{\tau-}}(-M_{\tau-}) = 1.$$

Therefore, (7) and (8) together imply

$$A_t = \int_0^{t\wedge\tau} h_s^2 ds = \int_0^{t\wedge\tau} 1_{[s\geq\tau_\alpha]}\frac{1}{M_{s-}^2}ds.$$

■

This theorem shows that under the market's information set, default is given by a totally inaccessible stopping time, generating a reduced form model from the market's perspective. We have an explicit representation of the intensity process as given by $\lambda_t = 1_{[t>\tau_\alpha]}\frac{1}{2[t-g(t-)]}$. The firm's default intensity is zero until time τ_α is reached. After time τ_α , the default intensity declines with the length of time that the firm remains in financial distress $(t-g(t-))$. The interpretation is that the longer the firm survives in the state of financial distress, the less likely it is to default. Presumably, the firm is more likely to recover and not reach the default magnitude of cash balances given by $2X_{\tau_\alpha}$. With this intensity, the market can value risky bonds and credit derivatives. This valuation is discussed in the next section.

4 Valuation of a Risky Zero-coupon Bond

Perhaps one of the most important uses of reduced form credit risk models is to price risky bonds and credit derivatives. This section studies the pricing of risky zero-coupon bonds. Let $(S_t)_{t\in[0,T]}$ denote the price process of a risky zero coupon bond issued by this firm that pays \$1 at time T if no default occurs prior to that date, and zero dollars otherwise. Then, under the no arbitrage assumption, S is given by

$$S_t = \mathbb{E}\left[\exp\left(-\int_t^T r_u du\right) 1_{[\tau_d>T]}\middle|\mathcal{G}_t\right],$$

where r_u is the instantaneous interest rate at time u, and $\mathbb{E}$ refers to the Risk Neutral probability law.

These "pure" risky zero coupon bonds are the building blocks for pricing coupon bonds and credit derivatives. Using the distribution equivalence between τ and τ_d (discussed in the previous section) we can rewrite this equation as:

$$S_t = \mathbb{E}\left[\exp\left(-\int_t^T r_u du\right) 1_{[\tau>T]}\middle| \mathcal{G}_t\right].$$

Lando [14] shows that this can be written as:

$$S_t = \mathbb{E}\left[\exp\left(-\int_t^T (r_u + \lambda_u) du\right)\middle| \mathcal{G}_t\right] \tag{10}$$

where the intensity process λ_u is as given in Theorem 1.

To facilitate the evaluation of expression (9), we will assume that interest rates are deterministic. (The inclusion of stochastic interest rates is straightforward, see Bielecki and Rutkowski [1].) In this case the valuation formula becomes

$$S_t = \exp\left(-\int_t^T r_u du\right)\mathbb{E}\left[\exp\left(-\int_t^T \lambda_u du\right)\middle| \mathcal{G}_t\right]. \tag{11}$$

Therefore, the price of a risky zero coupon bond at time $t = 0$ is given by evaluating the following expectation:

$$\mathbb{E}\left[\exp\left(-\int_0^T \lambda_u du\right)\right]. \tag{12}$$

The rest of this section is devoted to the computation of this expectation. Define $L_\alpha := \tau - g_{\tau_\alpha}$. L_α is the length of the first excursion of Brownian motion below zero exceeding length $\frac{\alpha^2}{2}$.

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\int_t^T \lambda_u du\right)\middle| \mathcal{G}_t\right] \\
=\ & 1_{[t<\tau]}1_{[t\geq\tau_\alpha]}\mathbb{E}\left[\exp\left(-\int_t^T \lambda_u du\right)\middle| \mathcal{G}_t\right] \\
&+1_{[t<\tau_\alpha]}\mathbb{E}\left[\exp\left(-\int_t^T \lambda_u du\right)\middle| \mathcal{G}_t\right] \\
=\ & 1_{[t<\tau]}1_{[t\geq\tau_\alpha]}\sqrt{t-g_t}\,\mathbb{E}\left[\frac{1}{\sqrt{\tau-g_t}}1_{[\tau\leq T]}\middle| \mathcal{G}_t\right] && (13)\\
&+1_{[t<\tau]}1_{[t\geq\tau_\alpha]}\frac{\sqrt{t-g_t}}{\sqrt{T-g_t}}\mathbb{E}\left[1_{[\tau>T]}\middle| \mathcal{G}_t\right] && (14)
\end{aligned}$$

$$+1_{[t<\tau_\alpha]}\frac{\alpha}{\sqrt{2}}\mathbb{E}\left[\frac{1}{\sqrt{\tau-g_{\tau_\alpha}}}1_{[\tau\leq T]}\middle|\mathcal{G}_t\right] \tag{15}$$

$$+1_{[t<\tau_\alpha]}\frac{\alpha}{\sqrt{2}}\mathbb{E}\left[\frac{1}{\sqrt{T-g_{\tau_\alpha}}}1_{[\tau_\alpha\leq T<\tau]}\middle|\mathcal{G}_t\right] \tag{16}$$

$$+1_{[t<\tau_\alpha]}\mathbb{E}\left[1_{[\tau_\alpha>T]}\middle|\mathcal{G}_t\right] \tag{17}$$

We next evaluate expressions (13)-(17). The distribution of the length of an excursion conditional on the age of the excursion is given in Chung [4].

Conditional expectation in (13) on the event $[\tau_\alpha \leq t < \tau]$:

$$\begin{aligned}\mathbb{E}\left[\frac{1}{\sqrt{\tau-g_t}}1_{[\tau\leq T]}\middle|\mathcal{G}_t\right] &= \mathbb{E}\left[\frac{1}{\sqrt{L^\alpha}}1_{[L^\alpha\leq T-g_t]}\middle|\mathcal{G}_t\right] \\ &= \int_{t-g_t}^{T-g_t}\frac{1}{2\sqrt{l}}\sqrt{\frac{t-g_t}{l^3}}dl \\ &= \frac{\sqrt{t-g_t}}{2}\left(\frac{1}{t-g_t}-\frac{1}{T-g_t}\right).\end{aligned}$$

Conditional expectation in (14) on the event $[\tau_\alpha \leq t < \tau]$:

$$\begin{aligned}\mathbb{E}\left[1_{[\tau>T]}\middle|\mathcal{G}_t\right] &= \mathbb{Q}\left[T-g_t<L^\alpha\middle|\mathcal{G}_t\right] \\ &= \int_{T-g_t}^{\infty}\frac{1}{2}\sqrt{\frac{t-g_t}{l^3}}dl \\ &= \sqrt{\frac{t-g_t}{T-g_t}}.\end{aligned}$$

Conditional expectation in (15) on the event $[\tau_\alpha > t]$:

$$\begin{aligned}\mathbb{E}\left[\frac{1}{\sqrt{\tau-g_{\tau_\alpha}}}1_{[\tau\leq T]}\middle|\mathcal{G}_t\right] &= \mathbb{E}\left[\mathbb{E}\left[\frac{1}{\sqrt{\tau-g_{\tau_\alpha}}}1_{[\tau\leq T]}\middle|\mathcal{G}_{\tau_\alpha}\right]\middle|\mathcal{G}_t\right] \\ &= \mathbb{E}\left[1_{[T\geq\tau_\alpha]}\left(\frac{1}{\alpha\sqrt{2}}-\frac{\alpha}{2\sqrt{2}}\frac{1}{T-g_{\tau_\alpha}}\right)\middle|\mathcal{G}_t\right].\end{aligned}$$

Conditional expectation in (16) on the event $[\tau_\alpha > t]$:

$$\begin{aligned}\mathbb{E}\left[\frac{1}{\sqrt{T-g_{\tau_\alpha}}}1_{[\tau_\alpha\leq T<\tau]}\middle|\mathcal{G}_t\right] &= \mathbb{E}\left[\mathbb{E}\left[\frac{1}{\sqrt{T-g_{\tau_\alpha}}}1_{[\tau_\alpha\leq T<\tau]}\middle|\mathcal{G}_{\tau_\alpha}\right]\middle|\mathcal{G}_t\right] \\ &= \frac{\alpha}{\sqrt{2}}\mathbb{E}\left[\frac{1}{T-g_{\tau_\alpha}}1_{[\tau_\alpha\leq T]}\middle|\mathcal{G}_t\right].\end{aligned}$$

In order to get the price, S_t, we need to obtain the law of τ_α on the event $[t<\tau_\alpha]$ conditional on $\mathcal{G}_t$. To find the Laplace transform of this density we introduce the following martingale as in Chesney, Jeanblanc-Picqué, Yor [3]

$$N_t := \Psi(-\lambda\mu_{t\wedge\tau_\alpha})\exp\left(-\frac{\lambda^2}{2}(t\wedge\tau_\alpha)\right),$$

where $\mu_t = \frac{M_t}{\sqrt{2}}$ and $\Psi(z) = \int_0^\infty x \exp\left(zx - \frac{x^2}{2}\right) dx$. Using the optional stopping theorem, we obtain

$$\mathbb{E}\left[\Psi(-\lambda\mu_{\tau_\alpha}) \exp\left(-\frac{\lambda^2}{2}(\tau_\alpha)\right)\middle| \mathcal{G}_t\right] = \Psi(-\lambda\mu_{t\wedge\tau_\alpha}) \exp\left(-\frac{\lambda^2}{2}(t \wedge \tau_\alpha)\right)$$

which in turn implies

$$1_{[t>\tau_\alpha]}\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\tau_\alpha\right)\middle| \mathcal{G}_t\right] = 1_{[t>\tau_\alpha]} \frac{\Psi(-\lambda\mu_t) \exp\left(-\frac{\lambda^2}{2}t\right)}{\Psi(\lambda\frac{\alpha}{\sqrt{2}})}.$$

Using relatively standard software, one can invert this Laplace transform and compute the expectations given in expressions (13)-(17).

For time 0, using expressions (13)-(17) in (11) gives:

$$S_0 = \exp\left(-\int_0^T r_u du\right)\left(1 - \frac{1}{2}\left(\mathbb{Q}\left[\tau_\alpha \leq T\right] - \frac{\alpha^2}{2}\mathbb{E}\left[\frac{1}{T - g_{\tau_\alpha}} 1_{[\tau_\alpha \leq T]}\right]\right)\right). \tag{18}$$

This is the price of the risky zero coupon bond at time 0. The interpretation of the last term in this expression is important. Default occurs not at time τ_α, but at time τ. The default time τ is, therefore, less likely than the hitting time τ_α. The probability $\mathbb{Q}\left[\tau_\alpha \leq T\right]$ is reduced to account for this difference.

Unfortunately, the law of τ_α is only known through its Laplace transform which is very difficult to invert analytically (see Chesney, Jeanblanc-Picqué, Yor [3] in this respect). Chesney, et al. gives the following formula.

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\tau_\alpha\right)\right] = \frac{1}{\Psi(\lambda\frac{\alpha}{\sqrt{2}})}.$$

Inverting this Laplace transform yields the law for τ_α, and given the law for τ_α, expression (18) is easily computed.

5 Conclusion

This paper provides an alternative method for generating reduced form credit risk models from structural models. The difference from Duffie and Lando [7] is that instead of using filtering theory to go from the manager's information to the market's as in Duffie and Lando, we use a reduction of the manager's information set. This modification is both conceptually and mathematically a different approach to the topic. Indeed, the perspective from filtering theory is that the market's information set is the same as the manager's, but with additional noise included. The perspective from reducing the manager's information set is that the market's information set is the same as the manager's, but the market just knows less of it. Future research agendas include investigating more complex structural models than those used herein, and more complex information reductions.

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